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Optimal Estimation of Local Orbit from GPS Measurements

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Introduction

UP to now, local or static solutions for the orbit determination problem have not been available due to the limited number of ground tracking stations. Notwithstanding, this lack of information has usually been overcome by using the knowledge of the orbit dynamics as in the dynamic optimal estimation methods[1] or in the preliminary orbit determination methods.[2] The recent appearance of the NAVSTAR concept of the Global Positioning System (GPS) brought about a fundamental change in the orbit determination problem since a point located on the Earth's surface is provided with a nearly continuous global world coverage of at least four GPS satellites simultaneously.[3,4] This provides a significant increase in the number of stations visible simultaneously by a target satellite. This fact suggests that the orbit determination methods may now search for statically determined solutions[5,6] and even for statically overdetermined solutions.[7] This Note presents an optimal solution to the problem of local orbit estimation of an Earth satellite from observations of range and range-rate provided by $n \geq 3$ distinct stations. This procedure is especially suitable for processing measurements from any redundant number of GPS satellites, unlike the statically determined solutions that select the best configuration to deal with the geometric dilution of precision (GDOP) phenomenon.[8] Expressions for the error covariance matrices are given too. First, the procedure evolves by assuming unbiased measurements, i.e., the clocks are assumed to be perfectly synchronized. Later, the bias is included with just a few modifications.

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Procedure Development

Let $n \geq 3$ be the number of tracking stations (or GPS satellites). Within a known inertial reference frame, the position and velocity vectors of the GPS stations and a target satellite are given, respectively, by R_i, V_i, $i = 1, \ldots, n$; r and v. Each station measures range y_i, and range-rate z_i is given by

$$y_i = |\rho_i| \qquad z_i = \rho_i' \, \nu_i / y_i \tag{1}$$

where $'$ indicates vector transposition, and ρ_i and ν_i are the position and velocity vectors of the satellite with respect to a station i, i.e.,

$$\rho_i = r - R_i \qquad \nu_i = v - V_i \tag{2}$$

The problem of orbit determination addressed here is that of seeking values for r and v from y_i, z_i, R_i, and V_i corresponding to the n stations. However, due to uncertainties in y_i, z_i, R_i, and V_i, Eqs. (1) and (2) may only be simultaneously satisfied in an approximate manner. The method proposed in this Note has been inspired by the algorithm QUEST, due to Shuster and Oh,[9] for estimating the attitude of satellites. It is based on minimizing the weighted summation of the squared residues of Eq. (2) separately, subjected to the constraints given by Eq. (1). The *position* estimation consists of solving the following problem of optimization:

minimize:

$$L(r, \rho_i) = \frac{1}{2} \Sigma \, a_i \, |r - (R_i + \rho_i)|^2 \tag{3}$$

subject to:

$$|\rho_i|^2 = y_i^2 \tag{4}$$

given y_i; R_i; a_i; $i = 1, \ldots, n \geq 3$, where a_i is a positive weight associated with the measurement taken from the ith station and satisfying $\Sigma \, a_i = 1$. One should point out that this first stage is self-contained and so it can be considered per se according to the interest of the application. A further stage of velocity estimation requires the previous knowledge of the vector r determined either from the former stage or by another method. In this way, the *velocity* is estimated so as to solve the following optimization problem:

minimize:

$$J(v, \nu_i) = \frac{1}{2} \Sigma \, b_i \, v - (V_i + \nu_i)|^2 \tag{5}$$

subject to:

$$\rho_i' \, \nu_i = y_i \, z_i \tag{6}$$

given z_i; y_i; ρ_i; V_i; b_i, $i = 1, \ldots, n \geq 3$, where b_i is the positive weight associated with the measurements taken from the ith station and satisfying $\Sigma \, b_i = 1$.

The solution of these two minimization problems by the Lagrange multipliers method is straightforward and gives[7]

$$r = \Sigma \, a_i (R_i + \rho_i) \tag{7}$$

$$\rho_i = y_i \, u_i \qquad u_i \equiv (r - R_i)/|r - R_i| \tag{8}$$

$$v = (\Sigma \, b_i \, u_i \, u_i')^{-1} \, \Sigma \, b_i \, u_i \, (u_i' \, V_i + z_i) \tag{9}$$

Substituting Eq. (7) in Eq. (8), the resulting equation may theoretically be solved with an arbitrarily high accuracy by the Newton-Raphson method, provided that the vectors u_i are not coplanar; i.e., the matrix $\Lambda(x_i) = \Sigma x_i \, u_i \, u_i'$ must have rank 3 for all positive x_i, $i = 1, \ldots, n$. Clearly, this condition also guarantees that Eq. (9) has a single solution.

Table 1 RMS of true errors (ϵ) and estimated standard deviation (σ): Numerical results for the Landsat orbit determination

			Unbiased Case				Biased Case			
	n	Stations	ϵ_r(m)	σ_r(m)	ϵ_v(m/s)	σ_v(m/s)	ϵ_r(m)	σ_r(m)	ϵ_v(m/s)	σ_v(m/s)
	5	1, 5, 6, 7, 8	36.7	36.8	0.278	0.276	58.9	54.2	0.277	0.276
Case	4	1, 5, 6, 7	38.1	39.1	0.293	0.293	73.8	74.1	0.294	0.293
A	4	1, 5, 6, 8	40.7	41.2	0.292	0.309	52.3	56.0	0.291	0.309
	3	1, 5, 8	49.1	46.8	0.355	0.351	61.8	60.8	0.355	0.351
	3[a]	6, 7, 8	108.5	101.4	0.733	0.760	203.0	198.0	0.746	0.760
	5	1, 5, 6, 7, 8	282.0	276.0	0.687	0.690	488.0	407.0	0.698	0.690
Case	4	1, 5, 6, 7	292.0	293.0	0.744	0.733	554.0	556.0	0.754	0.733
B	4	1, 5, 6, 8	314.0	309.0	0.744	0.772	401.0	419.0	0.741	0.772
	3	1, 5, 8	373.0	351.0	0.897	0.876	470.0	456.0	0.900	0.876
	3[a]	6, 7, 8	819.0	760.0	1.861	1.900	1539.0	1482.0	2.080	1.900

Case A: $\sigma_y = 10$ m; $\sigma_r = 16$ m; $\sigma_v = \sigma_z = 0.1$ m/s
Case B: $\sigma_y = \sigma_r = 100$ m; $\sigma_v = \sigma_z = 0.25$ m/s

[a]Close to alignment configuration.

Error covariance on the estimates of r and v can then be evaluated supposing R_i, V_i, y_i, and z_i are corrupted by unbiased uncorrelated random noises. Applying this model of small errors to Eqs. (7-9), and neglecting the effect of position uncertainties on velocity estimation, the following results:[7]

$$C_{rr} = \Lambda(a_i)^{-1}\Sigma a_i^2 u_i u_i'(\sigma_{Ri}^2 + \sigma_{yi}^2)\,\Lambda(a_i)^{-1} \tag{10}$$

$$C_{vv} = \Lambda(b_i)^{-1}\Sigma b_i^2 u_i u_i'(\sigma_{Vi}^2 + \sigma_{zi}^2)\,\Lambda(b_i)^{-1} \tag{11}$$

where $\sigma_{(.)}$ denotes the corresponding standard deviations. If one sets $a_i = \sigma_r^2/(\sigma_{Ri}^2 + \sigma_{yi}^2)$ and $b_i = \sigma_v^2/(\sigma_{Vi}^2 + \sigma_{zi}^2)$, these covariance expressions become

$$C_{rr} = \sigma_r^2\,\Lambda(a_i)^{-1} \qquad \sigma_r^2 = 1/\Sigma(\sigma_{Ri}^2 + \sigma_{yi}^2)^{-1} \tag{12}$$

$$C_{vv} = \sigma_v^2\,\Lambda(b_i)^{-1} \qquad \sigma_v^2 = 1/\Sigma(\sigma_{Vi}^2 + \sigma_{zi}^2)^{-1} \tag{13}$$

The error model also permits one to predict the performance indexes L and J at the solution. It can be shown that the expected values for $2L/\sigma_r^2$ and $2J/\sigma_v^2$ give the degrees of freedom of each problem, i.e., $n-3$. Therefore, one can check the validity of the error model by comparing directly these expected values with those computed from Eqs. (3) and (5).

One should remark that the closed-form solution for the position estimation and its error covariance matrix given, respectively, by Eqs. (7), (8), and (12) agree numerically with the results of the linearized least-squares approach.[10]

The procedure presented so far applies to the orbit determination problem with all the times involved exactly synchronized. However, if the target satellite clock is not perfectly synchronized or less accurate than the GPS clock, introducing an additional error source in range measurements, the basic idea of the proposed procedure still applies. One has only to express the old inputs y_i (range measurements) as

$$y_i = y_{pi} + \Delta y \tag{14}$$

where y_{pi} are the new inputs (pseudorange measurements) and Δy the bias due to time uncertainty, which is to be estimated together with the orbit parameters. The optimal estimation is accomplished by minimizing a modified performance index

$$L^*(r, \rho_i, \Delta y) = L(r, \rho_i) + \frac{1}{2}a^*\Delta y^2 \tag{15}$$

subjected to constraints (4) and (14), with the positive weights a^* and a_i obeying $a^* + \Sigma a_i = 1$. The estimate of bias is provided by adding one equation to the original ones

$$\Delta y = \Sigma a_i(|r - R_i| - y_{pi}) \tag{16}$$

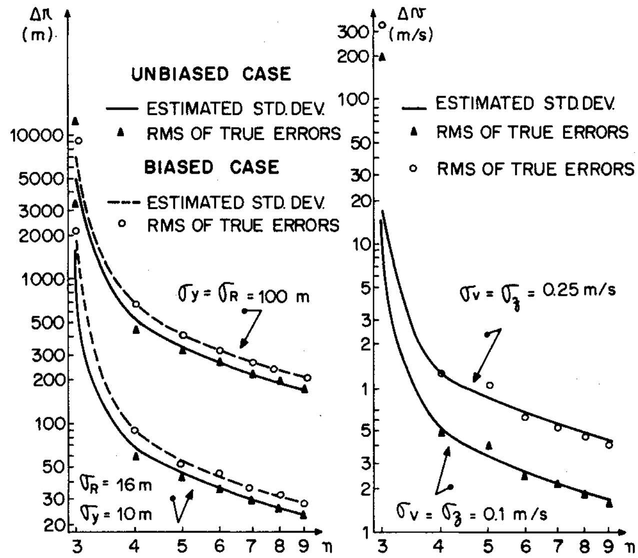

Fig. 1 Accuracy vs number of stations.

As in the synchronized case, one can set $a_i = \sigma_r^2/(\sigma_{Ri}^2 + \sigma_{yi}^2)$, $a^* = \sigma_r^2/\sigma_{\Delta y}^2$, where $\sigma_{\Delta y}$ is the range of uncertainty of bias with $\sigma_r^2 = [\sigma_{\Delta y}^{-2} + \Sigma(\sigma_{Ri}^2 + \sigma_{yi}^2)^{-1}]^{-1}$. So the new error covariances are now given by

$$C_{rr} = \sigma_r^2[\Lambda(a_i) - UU']^{-1} \qquad U \equiv \Sigma a_i u_i \tag{17}$$

$$C_{\Delta y \Delta y} = \sigma_r^2 + U' C_{rr} U \tag{18}$$

and both degrees of freedom on the position and velocity estimation equations remain unchanged.

Results

The algorithm developed was implemented to test mainly the procedure's preliminary performance. Some results are presented in order to indicate the potentiality of the method. In the first example, a configuration of five visible GPS satellites was considered to estimate the orbit of the Landsat D-satellite during 120 s. Table 1 summarizes these results. The aim of this first test was to illustrate typical expected results for a realistic situation. It was verified that the performance is worst when the stations are close to an alignment configuration (u_i coplanar).

A second test was performed by using n fictitious stations randomly located over a celestial hemisphere visible to a circular equatorial orbit satellite below them. The purpose of this second test was to verify the average impact on the orbit estimate accuracy when the number of stations is increased. The results are presented in Fig. 1. One should point out that the processing time did not present meaningful variations with the number of stations throughout the tests carried out, and the Newton-Raphson method converged in a single step to a high enough accuracy level.

Conclusions

A procedure for optimal orbit determination by processing single-time GPS measurements was presented. The error covariance matrices were obtained accounting not only for the measurement errors, but also for the uncertainties in the positions and velocities of the GPS satellites. Systematic errors due to nonsynchronized clocks were considered also. A very simple way of checking the statistical consistency of the error model was shown. The accuracy improves with the quantity of visible stations, and auxiliary algorithms to deal with the GDOP are not required. Preliminary results obtained through digital simulation indicate that the procedure is a promising tool for precision real-time orbit determination. Nevertheless, one hopes that its practical implementation will certainly give rise to future, more detailed studies than was possible in this introductory theoretical analysis.

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Calculation of Geodetic Coordinates from Earth-Centered, Earth-Fixed Coordinates

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Introduction

BOTH iterative and direct approximation methods exist for converting Earth-Centered, Earth-Fixed (ECEF) coordinates to geodetic coordinates. Of the iterative methods, Hedman[1] was shown to be very accurate and Lupash[2] was shown to be very efficient. Gersten's direct method[3] is also very fast and has been utilized in real-time applications since 1977.[4] This method is suitable when using 32-bit precision. The purpose of this Note is to present a simple refinement to Gersten's method which allows the achievement of greater accuracy when using 60-bit precision. The refinement consists of converting the approximate geodetic position back to ECEF coordinates, calculating the exact error and adjusting the geodetic position accordingly. A second adjustment should be made when using 120-bit precision. The Gersten and modified Gersten algorithms will be shown to be preferable to the other algorithms at precisions of 32, 60, and 120 bits.

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Gersten's Method

The derivation of Gersten's formula involves numerous approximations using binomial series expansions and back substitutions. Starting with ECEF coordinates (x,y,z), geodetic latitude is ultimately approximated by

$$\phi = \sin^{-1}(\sin\phi) \tag{1}$$

where

$$\sin\phi = \sin\phi^* \cos(\phi - \phi^*) + \cos\phi^* \sin(\phi - \phi^*) \tag{2}$$

$$\sin(\phi - \phi^*) = \sin\phi^* \cos\phi^* [1 + \epsilon - (2\epsilon - \tfrac{1}{2}e^2)\sin^2\phi^*] \tag{3}$$

$$\cos(\phi - \phi^*) = 1 - \tfrac{1}{2}\epsilon^2 \sin^2\phi^* \cos^2\phi^* \tag{4}$$

$$\sin\phi^* = z/r \tag{5}$$

$$\cos\phi^* = w/r \tag{6}$$

$$r = (x^2 + y^2 + z^2)^{1/2} \tag{7}$$

$$w = (x^2 + y^2)^{1/2} \tag{8}$$

$$\epsilon = ae^2/r \tag{9}$$

where a = length of ellipsoid semimajor axis, e_2 = ellipsoid eccentricity squared.

Geodetic height is approximated by

$$h = r - a + \tfrac{1}{2}ae^2 \sin^2\phi^* [1 + \epsilon - (\tfrac{1}{4}e^2 - \epsilon)\sin^2\phi^*] \tag{10}$$

and geodetic longitude is exactly calculated by $\lambda = \tan^{-1}(y/x)$.

When using 32-bit precision, the equation

$$\phi = \tan^{-1}(\sin\phi/\cos\phi) \tag{11}$$

where

$$\cos\phi = \cos\phi^* \cos(\phi - \phi^*) - \sin\phi^* \sin(\phi - \phi^*) \tag{12}$$

was found to be more accurate than Eq. (1). This requires setting $w = 1$ if $w < 1$ m. At 60- and 120-bit precision, Eq. (1) should be used.

Refinement

The inverse transformation (from geodetic to ECEF) is exact. Geodetic latitude ϕ, longitude λ and height h can be exactly converted to ECEF coordinates by

$$x = (r_e + h)\cos\phi\cos\lambda \tag{13a}$$

$$y = (r_e + h)\cos\phi\sin\lambda \tag{13b}$$

$$z = [(1 - e^2)r_e + h]\sin\phi \tag{13c}$$